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Computationally Efficient Optimization of Linear Time Invariant Systems using Haar wavelet

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***Abstract*— Optimization of linear time invariant system using existing methods is computationally costlier task for high order complex control systems. . Various types of transforms have been applied to overcome this issue and later on the problem of optimization has been solved with a special type of transform based on Haar wavelet, proving to be an excellent mathematical tool. The initial work on optimization has been done by Chen and Hsiao using recursive integral Haar operational matrix. In this paper, optimization of linear time invariant systems has been done using non-recursive integral operational matrix. The computational time taken by different methods has been shown using MATLAB 7.6.0.324 (R2008a) by graphs. It is observed that modified implantation takes lesser time as compared to the existing ones.**

***Index Terms*—Optimal control, time-invariant systems, Haar Wavelets, LQR**

I. INTRODUCTION

Optimal control problems with linear time invariant systems becomes difficult to solve when the order of system is sufficiently high. For higher order optimal control problems it is almost impossible to find out the optimal solution using analytical methods. Therefore the problems have been solved with the help of various types of numerical methods. The proceedings with the help of numerical methods contain large involvement of calculations and time. Therefore, many authors [1,2] proposed their methods to solve this kind of problem using different types of transforms. Haar wavelet based methods have been applied extensively in solving LQR optimal control problem because of its nice properties. The initial work has been pioneered by Hsiao [2,3,4] who first derived the various recursive Haar operational matrices for solving differential system equations[4]. The use of recursive Haar operational matrices in solving optimal control problems is computationally expensive task. Therefore a modified method has been proposed in this paper. In the proposed method the reverse integration matrix has been replaced by non-recursive integral operational matrix [5,6].

The rest of the paper is organised as follows. Section II describes the Haar wavelets. Integration operational matrices will be explained in section III. Detail of the algorithm will be discussed in Section IV. The algorithm has been demonstrated with the help of numerical examples in Section V. The paper is concluded in Section VI.

II. REVIEW OF HAAR WAVELET

The orthogonal set of Haar wavelet $h_i(t)$ is a group of square waves with magnitude of +1 and -1 in certain intervals and zero elsewhere[7,8]. The first curve is $h_0(t)$ which is known as father wavelet, the second curve $h_1(t)$ is the fundamental square wave which is also known as mother wavelet. By shifting and compressing of $h_1(t)$, one can obtain a complete orthogonal basis for the space of integrable functions over $t \in R$.
In other words $\{h_0(t), h(2^j t - k), j \in Z, k \in Z\}$ can span $\mathcal{L}^2$(R), where $h_0(t) = 1$, j is the dilation parameter and k is the translation parameter. For practical applications a signal is usually obtained from initial time up to some finite time. Thus, without loss of generality, the signal can be normalized to the time interval $t \in [0,1]$.under the assumption, the signal in the space $L^2[0,1]$ can be represented by the orthogonal basis $\{h_n, n \in Z^+\}$,
Where

$$h_0(t) = 1 \quad (1)$$
$$h_n(t) = h_1(2^j + k), \quad (2)$$
$$n = 2^j + k, j \geq 0, 0 \leq k \leq 2^j \quad (3)$$

Fig 1.1 is $h_0(t)=1$ during the whole interval $0 \leq t \leq 1$. It is the scaling function. The $h_1(t)$ is the mother wavelet and other subsequent curves are generated from it with two operations : translation and dilation

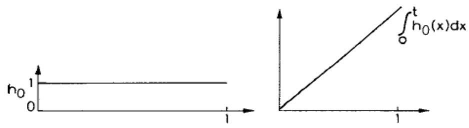

Fig 1.1 First Haar function and corresponding integral

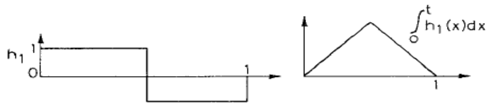

Fig 1.2 Second Haar function and corresponding integral

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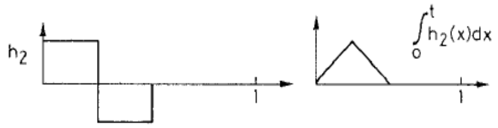

Fig 1.3 Third Haar function and corresponding integral

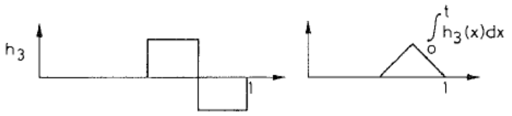

Fig 1.4 Fourth Haar function and corresponding integral

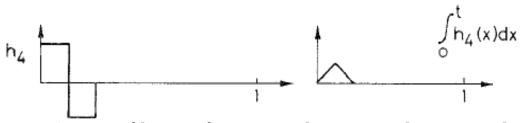

Fig 1.5 Fifth Haar function and corresponding integral

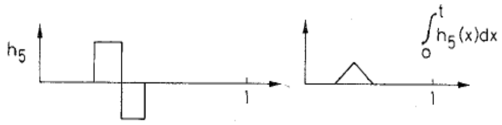

Fig 1.6 Sixth Haar function and corresponding integral

Fig 1.7 Seventh Haar function and corresponding integral

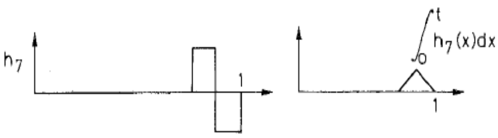

Fig 1.8 Eighth Haar function and corresponding integral

Any function *y*(*t*) which is square integrable in the interval [0,1) can be expanded in a Haar series :

$y(t)=c_0h_0(t)+c_1h_1(t)+.......= c_m^T H_m(t)$, $m = 2^j$, $j \geq 0$ (4)

$$c_i = 2^j \int_0^1 y(t)\, h_i(t)\, dt \qquad (5)$$

For *m*=4, $H_m(t)$ is calculated by taking the samples of first four Haar functions as follows:

$$H_4(t) \triangleq \begin{bmatrix} h_0(t) \\ h_1(t) \\ h_2(t) \\ h_3(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \qquad (6)$$

By matrix inversion as

$$c_m^T = y(t) H_m^{-1} \qquad (7)$$

Eq. (7) is called the *forward transform*, which turns the time function *y*(*t*) into the coefficient vector c^T and (4) is called the *inverse transform*, which recovers *y*(*t*) from c^T.

III. OPERATIONAL MATRICES FOR INTEGRATION

Recursive integration matrix-The Haar integration matrix [4] arises when integration is done forward in time from 1 to t where t>1. It can be expressed mathematically as

$$\int_1^t h_m(\tau) d\tau = P_m h_m(t) \qquad (8)$$

Where

$$P_m = \frac{1}{2m} \begin{bmatrix} 2mP_{\frac{m}{2}} & -H_{\frac{m}{2}} \\ H_{\frac{m}{2}}^{-1} & 0 \end{bmatrix} \qquad (9)$$

$P_1 = 1/2$

Non-recursive integration matrix:- The non recursive Haar integration matrix[5] is defined with the help of block pulse function as

$$P_m = h_m \{Q_{bm}\} h_m^{-1} \qquad (10)$$

Where,
P_m is Haar integration matrix of order m.
Q_{bm} is integration matrix for block pulse function[6]

$$Q_{B_m} = \frac{1}{m} \begin{bmatrix} \frac{1}{2} & 1 & \dots & 1 \\ 0 & \dots & \dots & \vdots \\ \vdots & 0 & \frac{1}{2} & 1 \\ 0 & \dots & 0 & \frac{1}{2} \end{bmatrix}$$

Block pulse function:- Over the time interval t ∈ (0, 1], we can find a complete and orthonormal set of block pulse functions[9], $\{\phi_i(t)\}$, i = 1,. . . , m, which is defined as follows:

$$\phi_i(t) = \begin{cases} \sqrt{m} & (i-1)/m < t \leq i/m \\ 0 & \text{elsewhere} \end{cases} \qquad (11)$$

IV. OPTIMAL CONTROL OF LINEAR TIME INVARIANT SYSTEMS VIA HAAR WAVELET

Consider a general LQR linear time invariant system with performance index J

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad (12)$$

$$J = \int_0^{t_f} [X^T(t) Q X(t) + u^T(t) R u(t)] dt \qquad (13)$$

Where A(t) is n x n matrix, B(t) is n x q matrix, Q is a positive semi definite n x n matrix and R is an r x r positive definite matrix.
For the solution of optimal control problem using Haar wavelets, the time interval [0, t_f) is normalized to [0, 1], therefore the normalized performance index can be written as

$$J=t_f\int_0^1[X^T(\tau)QX(\tau)+u^T(\tau)R(t)u(\tau)]d\tau \quad (14)$$

The adjoint state λ(τ), an n-vector, satisfies the following canonical equation [3]:

$$\begin{bmatrix}\dot{x}(\tau)\\ \dot{\lambda}(\tau)\end{bmatrix}=t_f\Omega(\tau)\begin{bmatrix}X(\tau)\\ \lambda(\tau)\end{bmatrix} \quad (15)$$

Where

$$\Omega(\tau)=\begin{bmatrix}A & -BR^{-1}B^T\\ -Q & -A^T\end{bmatrix} \quad (16)$$

the initial condition for x(0) and λ(1)=0

This equation is known as transversality condition. Let Φ(1,τ) be the state transition matrix of (15)

Decomposing Φ(1,τ) into the following form:

$$\Phi(1,\tau)=\begin{bmatrix}\varphi_{11}(1,\tau) & \varphi_{12}(1,\tau)\\ \varphi_{21}(1,\tau) & \varphi_{22}(1,\tau)\end{bmatrix} \quad (18)$$

We have

$$\begin{bmatrix}x(\tau)\\ \lambda(\tau)\end{bmatrix}=\begin{bmatrix}\varphi_{11}(1,\tau) & \varphi_{12}(1,\tau)\\ \varphi_{21}(1,\tau) & \varphi_{22}(1,\tau)\end{bmatrix}\begin{bmatrix}x_f\\ 0\end{bmatrix} \quad (19)$$

The transversality condition must hold leading to

$$\lambda(\tau)=\Phi_{21}(1,\tau)\Phi_{22}^{-1}(1,\tau)x(\tau) \quad (20)$$

The well known optimal control law is

$$u(\tau)=-R^{-1}(\tau)B^T(\tau)\lambda(\tau) \quad (21)$$

The feedback control law is

$$u(\tau)=-K(\tau)x(\tau) \quad (22)$$

from (20), (21) and (22)

$$K(\tau)=R^{-1}(\tau)B^T(\tau)\Phi_{21}(1,\tau)\Phi_{11}^{-1}(\tau) \quad (23)$$

Where Φ(1, 1)=*I*

In the Haar expansion, we expand $\dot{x}(\tau)$ and $\dot{\lambda}(\tau)$ into Haar series[3]

$$\begin{bmatrix}\dot{x}(\tau)\\ \dot{\lambda}(\tau)\end{bmatrix}=Ch(\tau) \quad (24)$$

Integrating both sides

$$\begin{bmatrix}x(\tau)\\ \lambda(\tau)\end{bmatrix}=\int Ch(\tau)$$

Introducing Haar integration operational matrix as

$$\begin{bmatrix}x(\tau)\\ \lambda(\tau)\end{bmatrix}=CPh(\tau)+\begin{bmatrix}x_f\\ 0\end{bmatrix} \quad (25)$$

Using (15) and (25)

$$C-\Omega(\tau)CP=[\,\Omega(\tau)\begin{bmatrix}x_f\\ 0\end{bmatrix}, 0_{2n}, \ldots, 0_{2n}] \quad (26)$$

In this manner by finding the value of Haar coefficient C, the columns of $\Phi_{11}(\tau)$ and $\Phi_{21}(\tau)$ can be evaluated. Finally the optimal feedback gain K(τ) is obtained from (23).
The above algorithm for optimal control of linear time-varying systems can be summarized in the following procedure.

Step 1: Normalize t∈ [0, t_f) to τ ∈ [0, 1).
Step 2: calculate the square matrix Ω from (16).
Step 3: Get P from (10)
Step 4: Compute C from (26)
Step 5: Calculate $\Phi_{11}(1,\tau)$ and $\Phi_{21}(1,\tau)$
Step 6: Finally obtain the optimal gain by (23)

A second order time invariant problem has been explained using modified algorithm in section V.

V. NUMERICAL EXAMPLE

Example of a second order linear time invariant system [5] for m=8.

Consider the state space equations as

$$\dot{x}(t)=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}x(t)+\begin{bmatrix}1\\ 0\end{bmatrix}u(t)$$

$$x(0)=\begin{bmatrix}0\\ 10\end{bmatrix} \quad x(t_f)\text{ unspecified} \quad (27)$$

The performance index of linear LQR problem is given as

$$J=\int_0^{\pi/2}[X^T(t)Q(t)X(t)+u^T(t)R(t)u(t)]dt \quad (32)$$

Where

$$Q(t)=\begin{bmatrix}0 & 0\\ 0 & 4\end{bmatrix} \quad (28)$$

$$R(t)=1 \quad (29)$$

The time-invariant optimal feedback gain K(t), t ∈[0, π/2), is required to be found.

Comparing with (12)

$$A(t)=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix} \quad B(t)=\begin{bmatrix}1\\ 0\end{bmatrix} \tag{30}$$

From (16) $\Omega(\tau)$ =

$$\begin{bmatrix} 0 & 0 & -\pi/2 & 0\\ \pi/2 & 0 & 0 & 0\\ 0 & 0 & 0 & -\pi/2\\ 0 & -2\pi & 0 & 0 \end{bmatrix} \tag{31}$$

From (10) P =

$$\begin{bmatrix} .5000 & -.2500 & -.1250 & -.0625 & .... & -.0625\\ .2500 & 0 & -.1250 & .1250 & .... & .0625\\ .0625 & .0625 & 0 & 0 & .... & 0\\ .0625 & -.0625 & 0000 & 0 & .... & .0625\\ \vdots & \vdots & \vdots & \vdots & .... & \vdots\\ .0156 & -.0156 & 0 & -.0313 & .... & 0 \end{bmatrix}$$

Following the steps given in algorithm, the solution of time-invariant optimal control problem has been obtained. The result obtained is shown graphically in Fig. 5.1 which satisfies with the existing analytical solution [5].

$$k_1= \frac{\sinh(\pi-2t)-\sin(\pi-2t)}{\cosh(\pi/2-t)^\wedge 2+\cos(\pi/2-t)^\wedge 2} \tag{32}$$

$$k_2= \frac{\cosh(\pi-2t)-\cos(\pi-2t)}{\cosh(\pi/2-t)^\wedge 2+\cos(\pi/2-t)^\wedge 2} \tag{33}$$

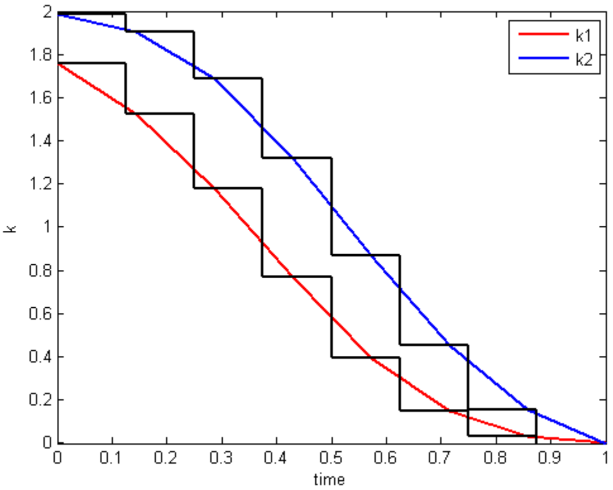

Fig 5.1 Optimal gain k versus time t

Example of a third order linear time invariant system [1] for m=16 & m=64

Consider the state space equations as

$$\dot{x}(t) = \begin{bmatrix}-1 & 0 & 0\\ 0 & 0 & -2\\ 0 & 2 & 0\end{bmatrix} x(t) + \begin{bmatrix}2\\ 2\\ 1\end{bmatrix} u(t) \tag{34}$$

$x(0)= x_0$ $\quad x(t_f)$ unspecified (35)

The performance index of linear LQR problem is given as

$$J=1/2 \int_0^2 [X^T(t)Q(t)X(t) + u^T(t)R(t)u(t)]dt \tag{36}$$

Where

$$Q(t) = \begin{bmatrix}2 & -2 & 0\\ -2 & 2 & 0\\ 0 & 0 & 0\end{bmatrix} \tag{37}$$

$$R(t)= 2 \tag{38}$$

The time-varying optimal feedback gain K(t), t ∈[0, 2), is required to be found.

Comparing with (12)

$$A=\begin{bmatrix}-1 & 0 & 0\\ 0 & 0 & -2\\ 0 & 2 & 0\end{bmatrix}, \quad B = \begin{bmatrix}2\\ 2\\ 1\end{bmatrix}$$

From (16) $\Omega(\tau)$ =

$$\begin{bmatrix} 2 & 0 & 0 & -4 & -4 & 2\\ 0 & 0 & 4 & -4 & -4 & 2\\ 0 & -4 & 0 & 2 & 2 & -1\\ -4 & 4 & 0 & -2 & 0 & 0\\ 4 & -4 & 0 & 0 & 0 & -4\\ 0 & 0 & 0 & 0 & 4 & 0 \end{bmatrix} \tag{39}$$

Following the steps given in algorithm the solution of problem has obtained. The solution obtained is shown graphically in Fig. 5.2. Analytical solution has not been reported in the literature [1].

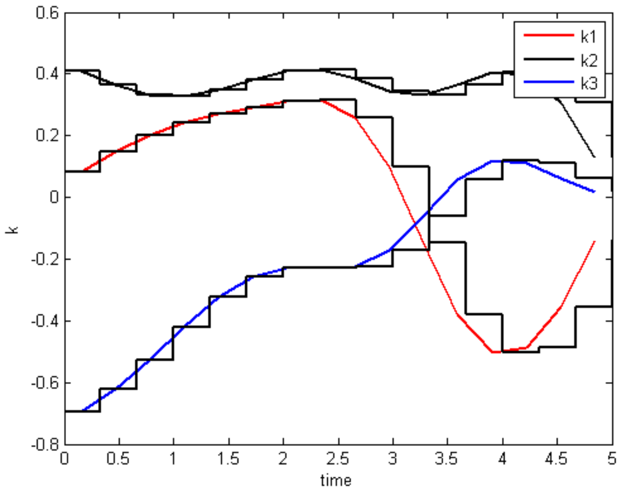

Fig 5.2 Optimal gain k versus time t

A graphical comparison between computational time involved in both the processes (recursive & nonrecursive) for different resolution is shown in Fig 5.3, by examining the graphs it is confirmed that modified method is computationally much efficient.

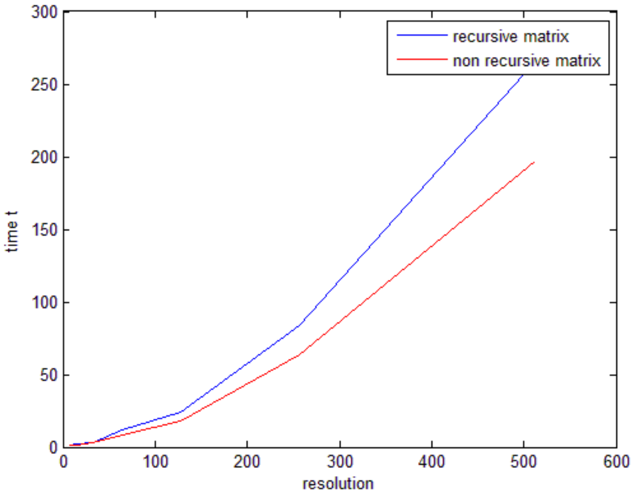

Fig 3 Time consumption t versus resolution m

VI. CONCLUSION

Non-recursive Haar integration operational matrix is applied successfully in optimal control problem. Computational efficiency of the non-recursive Haar integral operator matrix is demonstrated and compared with the corresponding recursive formulation, using the MATLAB. The proposed method can be applied to any linear time invariant problem of LQR system having any order. The results obtained are shown to agree well with respective reported results. The convergence is better at higher resolutions as shown by various plots. The improvement in computational efficiency, of the proposed non-recursive method as compared to the recursive technique, is established by comparing computation- times that takes to solve optimal control problems of complex system for different resolutions.

Future scope lies in solving other types of optimization problems using the proposed method.

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